Resit Exam Geometry - July 12, 2018

Note: This exam consists of **four** problems. Usage of the theory and examples of Chapters 1:1-5, 2:1-5, 3:1-3, 4:1-6 of Do Carmo's textbook is allowed. You may not use the results of the exercises, *with the exception of* the results of Exercise 1-5:2,12, 4-3:1,2. Give a precise reference to the theory and/or exercises you use for solving the problems.
You get 10 points for free.

All functions, curves, surfaces, parametrizations and (normal) vector fields in the exam problems are differentiable, i.e., of class C^∞.

Problem 1. (6+14=20 pt.)
Let $\alpha : I \to \mathbb{R}^3$ be a unit-speed curve with non-zero curvature at all points. Here I is an open interval in $\mathbb{R}$ with $0 \in I$. Let the Frenet frame at $\alpha(s)$ be $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$. Furthermore, $\mathbf{t}(s)$ makes a constant angle $\vartheta \in (0, \pi)$ with a fixed unit vector $\mathbf{u}$, for all $s \in I$. (So $\langle \mathbf{t}(s), \mathbf{u} \rangle = \cos\vartheta$ for all $s \in I$.)

Let C be the curve with parametrization $\gamma : I \to \mathbb{R}^3$ given by

$$\gamma(s) = \alpha(s) - \langle \alpha(s) - \alpha(0), \mathbf{u} \rangle \, \mathbf{u}.$$

1. Prove that C lies in the plane through $\alpha(0)$ orthogonal to $\mathbf{u}$.
2. Prove that the curvature of C at $\gamma(s)$ is $\dfrac{k(s)}{\sin^2\vartheta}$, where $k(s)$ is the curvature of α at $\alpha(s)$.

Problem 2. (8+6+8=22 pt.)
The goal of this problem is to prove the following theorem of Beltrami-Enneper:

Let C *be a regular asymptotic curve, with nowhere zero curvature, on a surface* S*. Then* $K = -\tau^2$ *at every point* p *of* C*, where* K *is the Gaussian curvature of* S *at* p *and* τ *is the torsion of the curve* C *(considered as a curve in* $\mathbb{R}^3$*) at* p*.*

A possible proof consists of the following steps. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame of the curve C at a point $p \in C$ (unique up to orientation of C).

1. Prove that $\{\mathbf{t}, \mathbf{n}\}$ is an orthonormal basis of T_pS.
2. As usual, let $dN_p : T_pS \to T_pS$ be the differential of the Gauss map at p. Prove that $dN_p(\mathbf{t}) = \pm\tau\,\mathbf{n}$ (the sign depends on the orientation of S).
3. Complete the proof of the theorem of Beltrami-Enneper.

   (Hint: Recall that dN_p is self-adjoint with respect to the inner product on T_pS.)

Assignments 3 and 4 on next page

Problem 3. (8+5+10=23 pt.)
Let $\alpha : I \to S$ be a unit-speed parametrization of a curve C on a regular surface S in $\mathbb{R}^3$. Here I is an open interval in $\mathbb{R}$. Let N be a (differentiable) unit normal field on S. The normal vector at the point $\alpha(s)$ is denoted by $N(s)$. The orthonormal frame consisting of the unit vectors $T(s) = \alpha'(s)$, $N(s)$ and $V(s) = N(s) \wedge T(s)$ is called the Darboux frame of the curve.

1. Prove that there are differentiable functions $k_n, \tau_g : I \to \mathbb{R}$ such that

$$N'(s) = -k_n(s)T(s) + \tau_g(s)V(s),$$

   where $k_n(s)$ is the normal curvature of the curve at $\alpha(s)$.

The function τ_g is called the geodesic torsion of the curve α.

2. Prove that C is a line of curvature of S if and only if $\tau_g(s) = 0$ for all $s \in I$.

3. Prove that C is both a line of curvature and an asymptotic curve of S if and only if it lies in a plane tangent to S at all points of C.

Problem 4. (8+9+8=25 pt.)
Let C be a regular curve (without self-intersections) in the half-plane $\{(x, 0, z) \mid x > 0\}$. Let S be the surface of revolution in $\mathbb{R}^3$ obtained by rotating C about the z-axis. It is convenient to use use the parametrization

$$\mathbf{x}(u, v) = (f(v)\cos u, f(v)\sin u, g(v)),$$

such that $v \mapsto (f(v), 0, g(v))$ is an arc-length parameterisation of C.

1. Let p and q be two points on C, and let $w_p \in T_pS$ be a unit tangent vector making an angle φ_0 with (the tangent line of) C at p. Let w_q be the vector obtained by parallel transporting w_p from p to q along C. Prove that w_q also makes an angle φ_0 with this meridian.

Consider a point $p \in C$. Let Γ be the parallel circle of S through p. Let $\overline{w}_p \in T_pS$ be the vector obtained by parallel transporting w_p once around Γ. Let ϑ_0 be the angle between T_pS and the horizontal plane through p.

2. Prove that the angle $\Delta\varphi$ between w_p and $\overline{w}_p$ is equal to $2\pi\cos\vartheta_0$.

3. Prove that $\Delta\varphi$ is zero if and only if Γ is a geodesic of S.

Solutions

Problem 1.

1. A straightforward computation shows that $\langle \gamma(s) - \alpha(0), \mathbf{u} \rangle = 0$. In other words, $\gamma(s)$ lies in the plane through $\alpha(0)$ with normal vector $\mathbf{u}$.

2. Note that

$$\gamma'(s) = \mathbf{t}(s) - \langle \mathbf{t}(s), \mathbf{u} \rangle \, \mathbf{u} = \mathbf{t}(s) - (\cos\vartheta)\, \mathbf{u},$$

so

$$|\gamma'(s)|^2 = |\mathbf{t}(s)|^2 - 2(\cos\vartheta)\langle \mathbf{t}(s), \mathbf{u} \rangle + (\cos^2\vartheta)|\mathbf{u}|^2 = \sin^2\vartheta.$$

Since $|\gamma'(s)| = \sin\vartheta > 0$, and a parametrization of C by arc length is given by $\beta(u) = \gamma((\sin\vartheta)^{-1}u)$. The curvature of C at $\gamma(s)$ is equal to the curvature of β at $\beta((\sin\vartheta)s)$. The curvature of β at $\beta(u)$ is

$$k_\beta(u) = |\beta''(u)| = \frac{|\gamma''((\sin\vartheta)^{-1}u)|}{\sin^2\vartheta}.$$

Since $\gamma''(s) = \mathbf{t}'(s) = k(s)\mathbf{n}(s)$ and $k(s) > 0$ for all $s \in I$ we get

$$k_\beta(u) = \frac{k((\sin\vartheta)^{-1}u)}{\sin^2\vartheta}.$$

So the curvature of C at $\gamma(s)$ is

$$k_\beta((\sin\vartheta)s) = \frac{k(s)}{\sin^2\vartheta}.$$

Remark: Part 2 can also be proven using the result of Exercise 1-5:12b.

Problem 2.

Let $\alpha : I \to S$, with $I = (-\varepsilon, \varepsilon)$, be a unit-speed parametrization of C with $\alpha(0) = p$ and $\alpha'(0) = \mathbf{t}$. Let N be the unit-normal field of S, and let $N(s)$ be the unit-normal at $\alpha(s)$.

1. Since C is an asymptotic curve, we have $\langle N(s), \alpha''(s) \rangle = 0$, for $s \in I$. Since $\alpha'(s) = \mathbf{t}(s)$, and, hence, $\alpha''(s) = k(s)\,\mathbf{n}(s)$, we see that $k(s)\langle N(s), \mathbf{n}(s) \rangle = 0$. Therefore, $\langle N(s), \mathbf{n}(s) \rangle = 0$, since $k(s) \neq 0$. So $\mathbf{n}(s) \in T_{\alpha(s)}S$. Since $\mathbf{t}(s) = \alpha'(s) \in T_{\alpha(s)}S$, we see that $\{\mathbf{t}(s), \mathbf{n}(s)\}$ is an orthonormal basis of $T_{\alpha(s)}S$.

2. The result of Part 1 implies that $N(s) = \pm\mathbf{b}(s)$. Therefore, $dN_p(\mathbf{t}) = N'(0) = \pm\mathbf{b}'(0) = \pm\tau\,\mathbf{n}$.

3. Let $dN_p(\mathbf{n}) = a\,\mathbf{t} + b\,\mathbf{n}$. Since dN_p is self-adjoint, we know that $a = \langle dN_p(\mathbf{n}), \mathbf{t} \rangle = \langle \mathbf{n}, dN_p(\mathbf{t}) \rangle = \pm\tau$. So the matrix of dN_p with respect to the basis $\{\mathbf{t}, \mathbf{n}\}$ of T_pS is

$$\begin{pmatrix} 0 & \pm\tau \\ \pm\tau & b \end{pmatrix}.$$

Hence, $K = \det dN_p = -\tau^2$.

Problem 3.

1. Since $\langle N', N\rangle = 0$, there are functions $a, b : I \to \mathbb{R}$ such that $N' = aT + bV$. Then $a = \langle N', T\rangle = -\langle N, T'\rangle = -\langle N, \alpha''\rangle = -k_n$. Renaming b to τ_g we get the desired identity.

2. This is a rephrasing of the Theorem of Olinde Rodrigues (Chapter 3-2, Proposition 3).

3. First assume that C lies in a plane tangent to S at every point of C. Then the normal of S along C is constant, so $N'(s) = 0$ for all $s \in I$. This implies that $k_n(s) = 0$ (so C is an asymptotic curve) and $\tau_g(s) = 0$ (so C is a line of curvature, according to Part 1).

Conversely, assume that $k_n(s) = 0$ and $\tau_g(s) = 0$, for all $s \in I$. Then $N'(s) = 0$, so $N(s) = N(s_0)$, for $s \in I$ ($s_0 \in I$ is arbitrary). Then C lies in the plane with normal $N(s_0)$ if and only if the function $f : I \to \mathbb{R}$, defined by $f(s) = \langle N(s_0), \alpha(s) - \alpha(0)\rangle$, is zero on I. Since $f(s_0) = 0$ and I is an interval, this follows from the fact that $f'(s) = \langle N(s_0), T(s)\rangle = \langle N(s), T(s)\rangle = 0$.

Problem 4.

1. The curve C is a meridian, and is therefore a geodesic of S (p. 255). An arc-length parameterisation is given by $\alpha(s) = (f(s), 0, g(s))$. Let w be the parallel vector field along α such that $w(s_0) = w_p$, and let φ be the angle between $\alpha'(s)$ and $w(s)$, so $\varphi(s_0) = \varphi_0$. Since α is a geodesic we have $\left[\frac{D\alpha'}{ds}\right] = 0$, and likewise, since w is parallel, $\left[\frac{Dw}{ds}\right] = 0$. Then it follows from Lemma 2 on p. 251 that $\frac{d\varphi}{ds} = 0$, and therefore $\varphi(s) = \varphi_0$ is a constant for all s.

2. Let $p = \mathbf{x}(u_0, v_0)$, and let $\beta\colon s \mapsto \mathbf{x}(u(s), v_0)$ be a unit-speed parameterisation of Γ. Since Γ is a circle of radius $r = f(v_0)$, it follows that $u(s) = s/r$. Since $\mathbf{x}$ is an orthogonal parameterisation, we may employ Proposition 3 on p. 252, which implies that

$$\frac{d\varphi}{ds} = -\frac{1}{2\sqrt{EG}}\left\{G_u\frac{dv}{ds} - E_v\frac{du}{ds}\right\}. \tag{1}$$

In our parameterisation we have

$$\begin{aligned}\mathbf{x}_u(u, v) &= (-f(v)\sin u, f(v)\cos u, 0)\\ \mathbf{x}_v(u, v) &= (f'(v)\cos u, f'(v)\sin u, g'(v)),\end{aligned}$$

which implies that

$$2E = f^2 \qquad F = 0 \qquad G = (f')^2 + (g')^2 = 1.$$

In particular, $E_v = 2ff'$, and $G_u = 0$, and Equation (1) becomes

$$\frac{d\varphi}{ds} = \frac{1}{2f}2ff'u' = f'(v_0)\frac{du}{ds} = \frac{f'(v_0)}{r} = \frac{\cos\vartheta_0}{r}.$$

Integrating, we find

$$\Delta\varphi = \int_0^{2\pi r}\frac{d\varphi}{ds}\,ds = 2\pi\cos\vartheta_0.$$

3. If Γ is a geodesic, then $\Delta\varphi = 0$. This is a direct consequence of Lemma 2 on p. 251, together with the definitions of parallel transport and of geodesics, and is not particular to our particular context. The converse assertion is not generally true, but it is in this case.

The previous question shows that $\Delta\varphi = 0$ if and only if $\cos\vartheta_0 = 0$. Since we have defined $\vartheta_0 \in (0, \pi)$, it follows that $\Delta\varphi = 0$ if and only if $\vartheta_0 = \frac{\pi}{2}$. This means that the tangent planes along Γ are parallel to the z-axis, and this is a necessary and sufficient condition for the parallel Γ to be a geodesic, as demonstrated on p. 256.